Rough Approximations via Maximal and Prime Ideals in Ring with Involution

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Abstract:

In this paper, we introduced the concept of the roughness of $*$ −maximal ideals with respect to the congruence relation induced by $*$ −maximal ideal of a $*$ − ring approximation space. Also, we discussed rough $*$ −prime ($*$−completely prime) ideals and presented several properties of $*$ −maximal and $*$ −prime ($*$−completely prime) ideals of a $*$ − ring approximation space.

Keywords: Rough set, rough maximal ideals, rough prime ideals, involution ring (*).

Introduction

The note of rough sets was introduced by Z. Pawlak [25] in 1982. Rough set theory is an extension of set theory. The main idea of rough sets corresponds to the concepts of lower and upper approximations of a set. Z. Pawlak [26] introduced the lower and upper approximations of a set with reference to an equivalence relation. Many mathematics were interested in studying the relationship between rough sets and algebra. For example, Z. Bonikowaski and Z. Pomykala [27, 11], studied Algebraic structures of rough sets, R. Biswas and S. Nanda [19], introduced the notion Rough Group and Rough subgroups, R. Chinram [18], studied the rough prime ideals and rough fuzzy prime ideals in $\Gamma-$ semigroups, N. Kuorki [13], introduced the notion of rough ideals of a semigroup,. Also construction of rough sets was studied in [3, 14], B. Davvaz [1], introduced the notion of Roughness in rings, B. Davvaz and Osman

Kazansi [2], introduced and studied the rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. And also, some properties of rough prime ideals were studied in [22, 17, 16], V. Selvan and G.Senthil Kumar [24], introduced the notion of rough ideals in semirings, Faraj. A. Abdunabi [7], introduced the notion of rough maximal ideals on ring approximation space. In addition, the concept of a ring with involution was studied in – depth by several mathematicians. As, I. N. Herstein [10] introduced the notion of rings with involution, R. Wiegandt [21], studied the structure of involution rings with chain condition, U. A. Aburawash and K. B. Sola [23], introduced the notion of $*-$Zero divisors and $*-$Prime ideals in ring with involution, E. Al Amin and K. B. Sola [5], introduced the notion of rough $*-$ ideals on $*-$ring approximation space.

In this paper, we present the concept of roughness of $*-$ maximal ideals. Also, we discuss rough $*$ −prime ($*$−completely prime) ideals and present several properties of these $*$−ideals of a $*$ − ring approximation space.

2. Approximations in ring

In this section, some well−known basic identities are given in Pawlak approximation space and ring approximation space.

Definition 2.1 [25]: A pair $(U, \sim)$ where $U \neq \emptyset$ and $\sim$ is an equivalence on U is called the *Pawlak approximation space.*

Definition 2.2 [26]: For an approximation space $(U, \sim)$ by a rough approximation in $(U, \sim)$ we mean a mapping $Apr: P(U) \to P(U) \times P(U)$: defined by:

$Apr\ (X) = (\underline{Apr}\ (X), \overline{Apr}\ (X))$ for every $X \in P(U)$; $X \subseteq U$, where

$\underline{Apr}\ (X) = \{x \in X\colon [x]_{\sim} \subseteq X\}$ and $\overline{Apr}\ (X) = \{x \in X\colon [x]_{\sim} \cap X \neq \emptyset\}$.

$\underline{Apr}$ (X) is called a lower rough approximation of X in (U, ~) where as $\overline{Apr}$ (X) is called upper rough approximation of X in (U, ~).

Definition 2.3 [26]: Given an approximation space $(U, \sim)$, a pair $(A, B) \in P(U) \times P(U)$ is called *a rough set* in $(U, \sim)$ iff $(A, B) = Apr\ (X)$, for some $X \in P\ (U)$.

In the category of rings, algebraic rings allow the study of algebraic structures that involve two primary operations: addition and multiplication on nonempty set. As defined in the following definitions.

Definition 2.4: A ring will always mean *an associative ring*, that is a nonempty set R together with two binary operations of + (addition) and . (multiplication) such that:

1. $(R, +)$ is an addition abelian group.

2. $(R, \cdot)$ is a multiplication semigroup.

3. A addition and multiplication are connected by the distributive laws; that is:

$a(b + c) = ab + ac$, and $(a + b)c = ac + ab$ for all $a, b, c \in R$.

Definition 2.5: A subset I of a ring R is called *a left (resp. right) ideal of* R if it satisfies the condition: $aI \subseteq I$ $(Ia \subseteq I)$ for $a \in R$. Clearly a left (resp. right) ideal of R. is *a subring of* R. A *two − sides ideals* I of a ring R (briefly called *an ideal* of R) is both a left and a right ideal of R, denoted by $I \lhd R$.

Definition 2.6: An ideal M of an involution ring R is called *maximal ideal* if $M \neq R$ and the only ideal strictly containing M is R.

Definition 2.7: An *ideal* P *of a ring* R is called *prime* if for any two ideals I, J of R the relation $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

Definition 2.8: An ideal P of a ring R is called completely− prime ideal of a ring R if for a, b $\in R$, $b \in P$ implies either $a \in P$ or $b \in P$.

Definition 2.9 [19]: an ideal defines an equivalence relation $\equiv_I$ of a ring R, given by: $a \equiv_I b \,(\mathrm{mod}\, I)$ iff $a - b \in I \;\forall\, a, b \in R$. we shall the pair $(R, \equiv_I)$ is called *a ring approximation space* where R is a ring and $\equiv_I$ is the relation induced by an ideal I of R and denote the set of all equivalence classes of elements of R under this relation by $R \,/\, I$ and will denote the equivalence class of an element *r* of R by $r/\, I$.

Proposition 2.10 [7]: let $(R$, mod $I)$ be a ring approximation space and I, J be two ideals of R Then:

1) $\overline{I(J)}$ and $\underline{I(J)}$ are rough ideals of R.

2) Let I is ideal and J is subring of ring R , Then $\overline{I(J)}$ and $\underline{I(J)}$ are rings.

Proposition 2.11[6]: If M is a maximal ideal of a ring R, For any $\mathrm{A} \subseteq \mathrm{R}$ and $\underline{M(A)} \cap \overline{M(A)}) \neq \emptyset$,then ($\underline{M(A)}$, $\overline{M(A)}$) is a rough maximal ideal of a ring R.

In the category of rings with involution, the involution is an anti−isomorphism of order 2 on R. It is evident that each commutative ring has at least one involution, namely the identical mapping [20]. So all homeomorphisms (embeddings) considered have to preserve involution and only subring which are closed under involution are admitted.

3.∗− ring approximation space.

In this section, we given some well–known basic identities in $*-$ ring approximation space. Also, we introduce the concept the $*-$ maximal ideals on a $*-$ ring approximation space and study some of its properties.

Definition 3.1: *An involution ring* R ($*$–ring) is an associative ring with additional unary $*$, called involution, subject to the familiar identities:

$$a^{**} = a,\ (ab)^* = b^*a^*,\ (a+b)^* = a^* + b^* \text{ for all } a,\ b \in R.$$

Definition 3.2: An ideal I of an involution ring R is called $*-$ *ideal*, and denoted by $I \lhd^* R$, if it is closed under involution; that is: $I^* = \{a^*;\ a \in I\} \subseteq I$.

Definition 3.4 [8]: An ideal M^* of an involution ring R is called $*-$ *maximal ideal* if $M^* \neq R^*$ and the only ideal strictly containing M^* is R^*.

Definition 3.5 [9]: An ideal P^* of an involution ring R is called $* -$ *prime ideal* if relation $IJ \subseteq P^*$ implies $I \subseteq P^*$ or $J \subseteq P^*$ where I, J are $* -$ *ideals* of R^*.

Definition 3.6 [23]: An ideal P^* of an involution ring R is called $* -$ *completely prime ideal* if for all $a, b \in R^*$, $ab \in P^*$ and $a^* b \in P^*$ implies either $a \in P^*$ or $b \in P^*$.

Proposition 3.7 [5]: let $(R^*, \text{mod } I^*)$ be a $*$ −ring approximation space and I^*, J^* be two $* -$ ideals of involution ring R^*, Then

(i) $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are rough $* -$ ideals of R^* .

(ii) Let I^* is $* -$ ideal and J^* is $*-$ subring of involution ring R^* , Then $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are an involution rings.

The following example shows Proposition 3.7.

Example 3.7: We consider the ring $R^* = Z_8$ and the ideal I = {0, 2, 4, 6}. we define the involution on R^* by $a^* = a \ \forall \ a \in Z_8$. There for a $*-$ ideal is {0, 2, 4, 6} and let X = {0, 1, 2, 4, 6}, Y = {1, 2, 3, 4, 5, 6, 7} Since the involution on Z_8 define by $a^* = a \ \forall a \in R^*$, Then $I^* = I$; I^* is a $*$ − ideal. For $x^* \in R^*$: $x^* + I^*$, it can get {0, 2, 4, 6}, {1, 3, 5, 7}. There for $\underline{I^*}(X) = \{x^* \in R^* : (x^* + I^*) \subseteq X\} = Z_8$ is a trivial $*$ − ideal Subsequently ideal in Z_8 and $\overline{I^*(X)} = \cup \{x^* \in R^* : (x^* + I^*) \cap X \neq \emptyset\} = \{0, 2, 4\}$ is $*$ – ideal in Z_8 Subsequently ideal in Z_8 ; I^* $(X) \subseteq (X)$; $I^* \subseteq$. So $\underline{I^*(X)}$ and $\overline{I^*(X)}$ are a rough $*-$ ideals. Thus rough ideal in Z_8 . Not that $\overline{I^*(X)}$is subring in Z_8 and is not ideal Now when because $7(2) = 6$ mod (8) and $6 \notin \overline{I^*(X)}$. There for $\underline{I^*(X)}$ and $\overline{I^*(X)}$ are an involution subrings of R^* and X is a rough an involution ring. Subsequently is a subring of R^* and X is a rough a ring. Now if Y = {1, 2, 3, 4, 5, 6, 7} then we have: $\underline{I^*(Y)} = \{1, 3, 5, 7\}$ and $\overline{I^*(Y)} = Z_8$ is a trivial $*-$ ideal . Not that $\underline{I^*(Y)}$ is $*-$ ideal in Z_8 Because $\underline{I^*(Y)}$ closed under involution; $\underline{I^*(Y)} = \underline{I(Y)}$. but not ideal because $\forall\, r \in Z_8 \wedge \forall\, a \in \underline{I(Y)}$ $\exists\, 0 \in Z_8 \wedge 3 \in \underline{I^*(Y)}$. So $0\,(3) = 0 \notin \underline{I^*(Y)}$; $\underline{I^*(Y)} = \underline{I(Y)}$. As well $\underline{I^*(Y)}$ is not subring because $3 - 1 = 2 \notin \underline{I^*(Y)}$. But $\underline{I^*(Y)}$ is not an involution sub ring in Z_8. There for $\underline{I^*(X)}$ and $\overline{I^*(X)}$ are not an involution subrings of Z_8. Subsequently are not a subring of Z_8 and Y is not a rough an involution ring. Therefor $\underline{I^*(X)}$ and $\overline{I^*(X)}$ is not a rough involution rings. Hence Y is not a rough a ring.

Definition 3.8: let $(R^*, mod\ M^*)$ be a $*-$ ring approximation space, M^* is an $*-$ Maximal ideal of involution ring R, *the upper and lower approximations of a* subset X of R^* *with respect of* M^* defined by:

$\overline{M^*(X)} = \{x^* \in R^* : (x + M^*) \cap X \neq \emptyset\}$, $\underline{M^*(X)} = \{x^* \in R^* : (x + M^*) \subseteq X\}$

Respectively. where $X \subseteq R^*$.

For the $*-$ ring approximation space $(R^*, mod\ M^*)$. The rough (undefinable) set can be expressed by its approximations with respect to M^* and written in the following form:

$Apr\ (X) = (\underline{M^*(X)}\ ,\ \overline{M^*(X)});\ \ X \subseteq R^*.$

The following example shows definition 3.8.

Examples 3.8: Let us the ring $R = Z_6$ and define the involution on R by $a^* = a\ \ \forall\ a \in R$. Suppose the $*$ −maximal ideal is {0, 2, 4} and X = {1, 2, 3, 4, 5}. Then $M^* = M$; M^* is a $*-$ ideal. For $x^* \in R$: $x^* + M^* = x + M$, we get $\{0^*, 2^*, 4^*\} = \{0, 2, 4\}$, $\{1^*, 3^*, 5^*\} = \{1, 3, 5\}$. So the upper approximation of X with respect of M^*as: $\overline{M^*(X)} = \cup \{x^* \in R^* :\ (x^* + M^*) \cap X \neq \emptyset\} = \overline{M(X)} = \{x \in R :\ (x + M) \cap X \neq \emptyset\} = \{0,2,4\} \cup \{1,3,5\} = \{0, 1, 2, 3, 4, 5\}$. And lower approximation of X with respect of M^* as: $\underline{M^*(X)} =$

$\{x^* \in R^* : (x^* + M^*) \subseteq X\} = \underline{M(X)} = \{x \in R : (x + M) \subseteq X\}$. So,

$\underline{M^*(X)} = \{1,3,5\}$, Based on that, are $\underline{M^*(X)}$ and $\overline{M^*(X)}$ a rough $*-$ maximal ideals of a involution ring R. Moreover, the boundary of X with respect of M^* is $BX = \overline{M^*(X)} - \underline{M^*(X)} = \{0,2,4\} \neq \emptyset$. Thus, X is rough set with respect of M^*.

Throughout this paper we use the $*-$ ring approximation space $(R^*, \equiv_{M^*})$ where R^* is a $*$–ring and $\equiv_{M^*}$is the relation induced by a $*$– maximal ideal M^* of R^*.

We can get the properties of $*$ –maximal ideal of involution ring R in the following Propositions:

Proposition 3.9: For a $*-$ ring approximation space $(R^*, m\ M^*)$, M^* is a $*-$ maximal ideal of an involution ring R and every subset $\mathrm{A}, \mathrm{B} \subseteq R^*$ we have:

1) $\underline{M^*(A)} \subseteq A \subseteq \overline{M^*(A)}$.

2) $\underline{M^*(\emptyset)} = \emptyset = \overline{M^*(\emptyset)}$,

3) $\underline{M^*(R^*)} = R^* = \overline{M^*(R^*)}$.

4) $\underline{M^*(A \cup B)} \supseteq \underline{M^*(A)} \cup \underline{M^*(B)}$.

5) $\underline{M^*(A \cap B)} = \underline{M^*(A)} \cap \underline{M^*(B)}$.

6) $\overline{M^*(A \cup B)} = \overline{M^*(A)} \cup \overline{M^*(B)}$.

7) $\overline{M^*(A \cap B)} \subseteq \overline{M^*(A)} \cap \overline{M^*(B)}$.

8) If $\mathrm{A} \subseteq \mathrm{B}$, then $M^*(\mathrm{A}) \subseteq M^*(\mathrm{B})$ and $\overline{M^*(A)} \subseteq \overline{M^*(B)}$

9) $\underline{M^*(\underline{M^*(A)}} = \overline{M^*\,(\underline{M^*(A)}} = \underline{M^*(A)}$.

10) $\overline{M^*(\overline{M^*(A)})} = \underline{M^*(\overline{M^*(A)})} = \overline{M^*(A)}$.

Proof: (1) Let $x^* \in \underline{M^*(A)}$; $\underline{M^*(A)} = \{x^* \in R^*: (x^* + M^*) \subseteq A\}$ then $x^* \in$

$x^* + M^* \subseteq A \implies \underline{M^*(A)} \subseteq A$. And so let $x^* \in A$ since $x^* \in x^* + M^*$ then

$x^* \in (x^* + M^*) \cap A \Longrightarrow (x^* + M^*) \cap A \neq \emptyset$. So $x^* \in \overline{M^*(A)}$; $\overline{M^*(A)} = \{x^* \in R^*: (x^* + M^*) \cap A \neq \emptyset\}$

Subsequently; $\underline{M^*(A)} \subseteq A \subseteq \overline{M^*(A)}$.

(2): $\underline{M^*(\emptyset)} = \{x^* \in R^*: (x^* + M^*) \subseteq \emptyset\} = \emptyset = \{x^* \in R^*: (x^* + M^*) \cap \emptyset = \emptyset\} = \overline{M^*(\emptyset)}$. And so $\underline{M^*(\emptyset)} = \emptyset = \overline{M^*(\emptyset)}$.

(3): It shows the way in at (1).

Since $\underline{M^*(R^*)} = \{x^* \in R^*: (x^* + M^*) \subseteq R^*\} = R^* = \{x^* \in R^*: (x^* + M^*) \cap R^* \neq \emptyset\} = \overline{M^*(R^*)}$

(4): Let $x^* \in \underline{M^*(A \cap B)} = \{x^* \in R^*: (x^* + M^*) \subseteq A \cap B\} \Leftrightarrow x^* + M^* \subseteq A \wedge x^* + M^* \subseteq B \Leftrightarrow x^* \in \underline{M^*(A)} \wedge x^* \in \underline{M^*(B)} \Leftrightarrow x^* \in \underline{M^*(A)} \cap \underline{M^*(B)}$.

(5): Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then $\underline{M^*(A)} \subseteq \underline{M^*(A \cup B)} \vee \underline{M^*(B)} \subseteq \underline{M^*(A \cup B)}$. So $\underline{M^*(A)} \cup \underline{M^*(B)} \subseteq \underline{M^*(A \cup B)}$.

(6) and (7): It shows the way in at (4) and (5) respectively.

(8): Let $x^* \in \overline{M^*(A)}$; $\overline{M^*(A)} = \{x^* \in R^*: (x^* + M^*) \cap A \neq \emptyset\}$ and Since $A \subseteq B$ then $(x^* + M^*) \cap B \neq \emptyset$. So $x^* \in \overline{M^*(B)}$; $\overline{M^*(B)} = \{x^* \in R^*: (x^* + M^*) \cap B \neq \emptyset\}$. Subsequently; $\overline{M^*(A)} \subseteq \overline{M^*(B)}$. In a similar way we can prove $\underline{M^*(A)} \subseteq \underline{M^*(B)}$.

(9): $\underline{M^*(\underline{M^*(A)})} = \left\{x^* \in R^* : (x^* + M^*) \subseteq \underline{M^*(A)}\right\} = \{x^* \in R^*: (x^* + M^*) \subseteq \{x^* \in R^*: (x^* + M^*) \subseteq A\}\} = \{x^* \in R^*: (x^* + M^*) \subseteq A\} = \underline{M^*(A)}$. And so $\overline{M^*(\underline{M^*(A)})} = \left\{x^* \in R^*: (x^* + M^*) \cap \underline{M^*(A)} \neq \emptyset\right\} = \{x^* \in R^*: (x^* + M^*) \cap \{x^* \in R^*: (x^* + M^*) \subseteq A\} \neq \emptyset\}$

$= \{x^* \in R^*: (x^* + M^*) \subseteq A\} = \underline{M^*(A)}$.

(10): It shows the way in at (9)■

Proposition 3.10: let $(R^*, \text{mod } M^*)$ be a $*$ −ring approximation space and M^* is a $*$ − maximal ideal of an involution ring R, For any $A \subseteq R^*$, the following hold:

1) $\underline{M^*(R^*/A)} = {}^{R^*}\!\Big/\overline{M^*(A)}$; 2) $\overline{M^*(R^*/A)} = {}^{R^*}\!\Big/\underline{M^*(A)}$;

3) $\underline{M^*(A)} = (\overline{M^*(A^c}\,)^c$; 4) $\overline{M^*(A)} = (\underline{M^*(A^c)}\,)^c$.

Proof: By using definition the upper and lower approximation of A with

respect of M^* we find that (1): $\underline{M^*(R^*/A)} = \{x^* \in R^*: (x^* + M^*) \subseteq R^*/A\} =$

$\{x^* \in R^*: (x^* + M^*) \subseteq A^c\} \Longrightarrow (x^* + M^*) \nsubseteq A \Longrightarrow A \subseteq (x^* + M^*)^c$. So

${}^{R^*}\!\Big/\overline{M^*(A)} = \{x^* \in R^*: A \subseteq (x^* + M^*)^c\} =$

$R^* - \{x^* \in R^*: (x^* + M^*) \subseteq A\}$. Subsequently $\underline{M^*(R^*/A)} = {}^{R^*}\!\Big/\overline{M^*(A)}$.

(2): It shows the way in at (1).

(3): $\underline{M^*(A)} = \{x^* \in R^*: (x^* + M^*) \subseteq A\} = \{x^* \in R^*: (x^* + M^*) \subseteq (A^c)^c\} =$

$\underline{M^*((A^c)^c)} = \{x^* \in R^*: (x^* + M^*) \cap (A^c)^c \neq \emptyset\} = (\overline{M^*(A^c}\,)^c$.

(4): It shows the way in at (3)■

Proposition 3.11: let (R*, mod M^*) be a $*$ −ring approximation space and M^* is $*$ − maximal ideal of a involution ring R and A, B are non−empty subset of R*then:

1) $\overline{M^*(A)} + \overline{M^*(B)} = \overline{M^*(A+B)}$; $\underline{M^*(A)} + \underline{M^*(B)} \subseteq \underline{M^*(A+B)}$

2) $\overline{M^*(A.B)} = \overline{M^*(A)}.\overline{M^*(B)}$; $\underline{M^*(A)}\,.\underline{M^*(B)} \subseteq \underline{M^*(A\,.B)}$

The following example shows Proposition 3.11.

Example 3.11: Let us the ring $R^* = Z_4$ and define the involution on Z_4 by $a^* = a \quad \forall\, a \in Z_4$. Suppose the $*$ − maximal ideal is {0, 2} and *A* = {1, 2, 3}, *B* = {0, 1} *are non−empty subset of* R^* .

Solution: Since the involution on R define by $a^* = a \quad \forall\, a \in Z_4$, Then M^* $= M$; M^* is a $*$ − ideal. For $x^* \in R^*$: $x^* + M^* = x + M$, we get {0*, 2*} = {0, 2}. So the upper approximation of A, B with respect of M^* as: $\overline{M^*(A)} = \cup$ $\{x^* \in R^* : \ (x^* + M^*) \cap A \neq \emptyset\} = \overline{M(A)} = \{x^* \in R^* : \ (x + M) \cap A \neq \emptyset\} =$

$\{0,2\} \cup \{1,3\} = \{0,1,2,3\} = \overline{M(B)}.$ $as\ A + B = \sum_{i=1}^{n}(a_i + b_i)$; $a_i \in A, b_i \in B$ then $\overline{M^*(A)} + \overline{M^*(B)} = \{0,1,2,3\} = \overline{M^*(A+B)}$. $As\ AB =$

$\sum_{i=1}^{n}(a_i b_i)$; $a_i \in A,\ b_i \in B$ = $\{0,1,2,3\}$ = $\overline{M^*(AB)} = \cup \{x^* \in R^* :$

$(x^* + M^*) \cap AB \neq \emptyset\}$ = $\overline{M(A)}$. $\overline{M(B)}$. And also $\underline{M^*(A)} =$

$\{1,3\}$, $\underline{M^*(B)} = \emptyset$ then $\underline{M^*(A)}.\underline{M^*(B)} = \emptyset$. Since $\underline{M^*(A.B)} =$

$\{x^* \in R^* :\ (x + M^*) \subseteq A.B\} = \{0,1,2,3\}$. Hence $\underline{M^*(A)}.\underline{M^*(B)} \subseteq$

$\underline{M^*(A.B)}$, And also $\underline{M^*(A)} + \underline{M^*(B)} = \emptyset \subseteq \{0,1,2,3\} = \underline{M^*(A+B)}$.

4. Roughness of $*-$ maximal ideals in $*-$rings.

In this section, we introduce the concept of rough $*-$ maximal ideals on a $*-$ ring approximation space and give some results on them.

Proposition 4.1: let (R^*, mod M^*) be a $*$ −ring approximation space and M^* be a $*-$ ideal of a involution ring R. If M^* is $*$ − maximal ideal of a involution ring R then $\overline{M^*(A)}$) $and\ \underline{M^*(A)}$ are is a $*$ −maximal ideals of a involution ring R.

Proof: As M^* is an $*-$ ideal of a involution ring R then $M^* = M;\ M$ is an ideal of a ring R and $x^* + M^* = x + M \subseteq A$. There for $(x^* + M^*) \cap A = (x + M) \cap A \neq \emptyset$. Hence $\underline{M^*(A)} = \underline{M(A)}$ and $\overline{M^*(A)} = \overline{M(A)}$.

Subsequently $\overline{M^*(A)}$) $and\ \underline{M^*(A)}$ are a $*$ − ideals of an involution ring R ■

Proposition 4.2: let (R^*, mod M^*) be a $*$ −ring approximation space. If M^* is a $*$ − maximal ideal of a involution ring R, For any subset $A \subseteq R^*$ and $\underline{M^*(A)} \neq \emptyset$, then $\overline{M^*(A)}$) $and\ \underline{M^*(A)}$ are a rough $*$ − maximal ideals of a involution ring R.

Proof: As M^* is a is $*-$ maximal ideal of a involution ring R then $M^* = M \subseteq R$ is $*$−maximal ideal. And as $x^* + M^* = x + M$ and $A \subseteq R^*$ then $(x^* + M^*) \cap A = (x + M) \cap R^* \neq \emptyset$. Hence $\overline{M^*(A)}$ is the upper approximation of a subset A of R^* with respect M^*. From proposition (4.1) we have $\overline{M^*(A)}$) *and* $\underline{M^*(A)}$ are $*-$ ideals of a involution ring R . Since $x^* + M^* = x^* + M \subseteq A$ then $\underline{M^*(A)}$ is the lower approximation of a subset A of $R*$ with respect M^*. Hence $\overline{M^*(A)}$) *and* $\underline{M^*(A)}$ are rough maximal $*-$ ideals of involution ring R ■

5. Rough $*-$ prime and $*$−completely prime ideals in $*-$ring

In this section, we discuss the rough prime and rough $*-$ completely prime ideals on a $*-$ ring approximation space and give some properties of such $*-$ideals.

Proposition 5.1: let $(\mathrm{R}^*$, mod $I^*)$ be a $*-$ring approximation space. If P^* is a $*-$prime ideal $of\ a\ involution\ ring$ R such that $\underline{I^*(P^*)} \neq \emptyset$, then $\underline{I^*(P^*)}$ and $\overline{I^*(P^*)}$ are a rough $*-$ prime ideals of R^*.

Proof: Since $I^* \subseteq I\ and\ P^* \subseteq P\ \ then\ x^* + I^* \subseteq x + I$ and also $(x^* + I^* \subseteq P^*) \subseteq (x + I \subseteq P)$. Subsequently $\underline{I^*(P^*)} \subseteq \underline{I(P)}$. And since $x^* + I^* \cap P^* \subseteq P^* \subseteq x + I \cap \mathrm{P} \neq \emptyset$; $x \in R,\ I \lhd R$ then $\overline{I^*(P^*)} \subseteq \overline{I(P)}$. So $\underline{I^*(P^*)}$ and $\overline{I^*(P^*)}$ are $*-$ideals of R^* and are a lower and upper approximations of a $*-$ideal I^* with respect P^* respectively. Now let $\underline{J^*(P^*)}\ \underline{Q^*(P^*)} \subseteq \underline{I^*(P^*)}$. As P^* is a $*-$prime ideal of a involution ring R then $J^* \subseteq P^*$ or $Q^* \subseteq P^*$; $J^*, Q^* \lhd^* R^*$. So $x^* + J^* \subseteq P^*$ or $y^* + Q^* \subseteq P^*$ for all $x^*, y^* \in R^*$. Hence $\underline{J^*(P^*)} \subseteq \underline{I^*(P^*)}$ or $\underline{Q^*(P^*)} \subseteq$. Thus

$(\underline{I^*(P^*)}$ $is\ a$ $*-$ prime ideal of R^*. As $(x^* + J^*) \cap P^* \subseteq P^* \neq \emptyset$ or $\subseteq$

$(y^* + Q^*) \cap P^* \subseteq P^* \neq \emptyset$ then $\overline{J^*(P^*)} \subseteq \overline{I^*(P^*)}$ or $\overline{Q^*(P^*)} \subseteq \overline{I^*(P^*)}$.

Hence $\overline{I^*(P^*)}$ $is\ a$ $*-$ prime ideal of R^*. Hence $\underline{I^*(P^*)}$ and $\overline{I^*(P^*)}$ are a rough $*-$prime ideals of R^* ■

Proposition 5.2: let $(R^*$, mod $I^*)$ be a $*$ −ring approximation space. If I^* rough $*$ − ideal and P^* is a rough$*-$ prime ideal of an involution ring R, then $P^* \cap I^*$ is a rough $*$ − ideal of R^*.

Proof: Since $\underline{(P^* \cap I^*)(A)}$ is closed under involution then

$\underline{(P^* \cap I^*)(A)}$ is a $*$ − ideals. And Since I^* is rough $*-$ ideal and P^* is a rough $*$ − prime ideal of an involution ring R. Then For any $A \subseteq R^*$ $\underline{I^*(A)}$ and $\overline{I^*(A)}$) are rough $*-$ ideals, also $\underline{P^*(A)}$ and $\overline{P^*(A)}$ are rough prime $*$ −

ideals, respectively. Now, Since $P^* \cap I^* \subseteq I^*$ and $P^* \cap I^* \subseteq P^*$ then $P^* \cap I^*$ is a $*$ −ideal of R^*. by proposition 3.9. we have $\underline{(P^* \cap I^*)(A)} = \underline{P^*(A)} \cap \underline{I^*(A)} = \{x^* \in R^* : (x^* + P^*) \subseteq A\} \cap \{x^* \in R^* : (x^* + \mathrm{I}^*) \subseteq A\} = \{x^* \in R^* : (x^* + (P^* \cap I^*)) \subseteq A\} = \underline{(P^* \cap I^*)(A)}$. So $\underline{(P^* \cap I^*)(A)}$ is a lower approximation of a subset A of with respect $P^* \cap I^*$. And also As $(P^* \cap I^*) \subseteq (P \cap I)$then $(P^* \cap I^*) \cap A \subseteq (P \cap I) \cap A \neq \emptyset$. Therefore $\overline{(P^* \cap I^*)(A)}$ is a upper approximation of a subset A of R^* with respect $P^* \cap I^*$. Hence $(\underline{P^* \cap I^*)(A)}$ and $\overline{(P^* \cap I^*)(A)}$ are a rough $*$− ideals of a involution ring R ■

Proposition 5.3: let $(\mathrm{R}^*$, mod $M^*)$ be a $*$ −ring approximation space then every rough a $*$−maximal ideal is rough $*$−prime ideal in a $*$−ring.

Proof: Let ($\underline{M^*(A)}$, $\overline{M^*(A)}$) is rough a $*$−maximal ideal of R^*. If M^* is maximal ideal, then M^* is a prime ideal. There for M^* is $*$−prime ideal [20]. Hence ($\underline{M^*(A)}$, $\overline{M^*(A)}$) is rough $*$ − prime ideal of involution ring R■

Proposition 5.4: let (R^*, mod I^*) be a $*$ −ring approximation space. If P^* is a $*$ −completely prim ideal of R^*such that such that $\underline{I^*(P^*)} \neq \emptyset$, then $\underline{I^*(P^*)}$ *and* $\overline{I^*(P^*)}$ is $*$−completely prim ideal of R^*.

Proof: Let x y and $x^*y \in \overline{I^*(P^*)}$ then $x\,y + I^* \cap P^* \neq \emptyset$ and $x^*y + I^* \cap P^* \neq \emptyset$. Since $\big((x+I^*)(y+I)\big) \cap P^* \neq \emptyset$. So $\exists$ a b $\in (x^*y+I^*) \cap P^*$. Thus $a \in (x^*+I^*) \cap P^*$ or $b \in (y+I^*) \cap P^*$. Since P^* is a $*$−completely prime ideal of R^* then either a $\in P^*$ or b $\in P^*$. So $a \in (x+I^*) \cap P^* \neq \emptyset$ or $b \in (y+I^*) \cap P^* \neq \emptyset$. Therefor either $x \in \overline{I^*(P^*)}$ or $y \in \overline{I^*(P^*)}$. Hence $\overline{I^*(P^*)}$ is $*$ −completely prime ideal of R^*. Now we prove $\underline{I^*(P^*)}$ is $*$ −completely prim ideal of R^*. Now let $x\ y$ and x^*y

$\in \underline{I^*(P^*)}$, then $(xy + I^*) \subseteq P^*$ and $(x^*y + I^*) \subseteq P^*$, it follows that $\big((x + I^*)(y + I^*)\big) \subseteq P^*$. Now le t $(x + I^*) \subseteq P^*$ and $y + I^* \subseteq P^*$. Then there exists a $\in x + I^*$ and b $\in y + I^*$. As P^* is a $*$ $-$completely prime ideal of R^* then either $(x + I^*) \subseteq P^*$, or $y + I^* \subseteq P^*$. Which is a contradiction. Therefore, either $x \in \underline{I^*(P^*)}$,

Or $y \in \underline{I^*(P^*)}$. From proposition 5.1 $\overline{I^*(P^*)}$ and $\underline{I^*(P^*)}$ are an $*$ $-$ ideal of R^*. Hence $\underline{I^*(P^*)}$, is $*$ $-$completely prime ideal of R^* ■

Proposition 5.5: let $(R^*$, mod $I^*)$ be a $*$ $-$ring approximation space. If P_i^* *a set* rough $*$ $-$completely prime ideals of an involution ring R, then $(\bigcup_{i\in I} {P^*}_i)$ is a rough $*$ $-$ completely prime ideal of an involution ring R.

Proof: Since $x^* \in I^* \subseteq x^* + I^*$ and $P_i^* \subseteq \bigcup_{i\in I} {P^*}_i$ for some $i \in I$. Then $x^* + I^* \cap P_i^* \subseteq x^* + I^* \cap \bigcup_{i\in I} {P^*}_i \neq \emptyset$. Hence $\overline{I^*(\bigcup_{i\in I} {P^*}_i)}$ is a upper approximation of a $*$ $-$ideal I^* with respect $\bigcup_{i\in I} {P^*}_i$. As P_i^* rough $*$

$-$completely prime ideals of an involution ring R then $x^* + I^* \subseteq P_i^* \subseteq \bigcup_{i\in I} P^*_i$ for some $i \in I$. Thus $x^* + I^* \subseteq \bigcup_{i\in I} P^*_i$ Hence $\overline{I^*(\bigcup_{i\in I} P^*_i)}$ is a lower approximation of a $*-$ideal I^* with respect $\bigcup_{i\in I} P^*_i$. Now, let $xy \in \bigcup_{i\in I} P^*_i$ and $x^*y \in \bigcup_{i\in I} P^*_i$, by proposition , we have that $\overline{I^*(\bigcup_{i\in I} P^*_i)} = \bigcup_{i\in I} \overline{I^*(P^*_i)}$. Thus $xy \in \overline{I^*(P^*_i)}$ and $x^*y \in \overline{I^*(P^*_i)}$ for some $i \in I$. As $\overline{I^*(P^*_i)}$ is $*-$prime ideals of an involution ring R, that is, $x \in P^*_i$ or $y \in P^*_i$. Thus, $x \in \bigcup_{i\in I} P^*_i$ or $y \in \bigcup_{i\in I} P^*_i$. Hence $\bigcup_{i\in I} P^*_i$ a $*-$ rough completely prime ideal of an involution ring R■

Example 5.6: Let a ring $R^* = Z$ and the involution is defined by $a^* = a$ $\forall\ a \in$. Let $I^* = (6)$ is a $*-$ ideal, $P^*_i = (3), (5), (7)$; $i \in I = \{1, 2, 3\}$. It is clearly $I^* = I$ and $P^*_i = P_i\ \forall\ i \in I$. For $x^* \in R^*$: $x^* + I^* = x + I$ then $x^* + (6) \nsubseteq \bigcap_{i\in I} P_i^* = \emptyset$ and $x^* + (6) \nsubseteq P^*_i$; $i \in I$. Also $x^* + (6) \cap (\bigcup_{i\in I} P^*_i) \neq \emptyset$ and $x^* + (6) \cap P^*_i \neq \emptyset$. Thus $\overline{I^*(\bigcup_{i\in I} P^*_i)} = \bigcup_{i\in I} \overline{I^*(P^*_i)}$ and

$\underline{I^*(\bigcap_{i\in I} P_i^{\,*})} \neq \bigcap_{i\in I} \underline{I^*(P_i^{\,*})}$. In general, $\bigcap_{i\in I} P_i^{\,*}$ is not a rough $*$ − completely prime ideal of R^*.

Proposition 5.7: let $(R^*, \text{mod } I^*)$be a $*$ −ring approximation space. If P^* is a $*$−completely prime ideal of R^* such that such that $\underline{I^*(P^*)} \neq \emptyset$, then $(\underline{I^*(P^*)}, \overline{I^*(P^*)})$ is a rough $*$−completely prime ideal of R^*.

Proof: This follows from Propositions 5.1and 5.4.

Conclusion:

In this paper, our study of $*$− maximal and $*$ − prime ideals provide valuable insights into the structure of $*$−ring approximation space. The properties we discussed contribute to enhancing our understanding of these ideals and open new avenues for future research in the theory of $*$−rings.

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الملخص:

في هذه الورقة قدمنا مفهوم الخشونة للمثاليات الالتفافية الاعظمية بالنسبة لعلاقة التكافؤ المولدة بواسطة المثالي الالتفافي الاعظمي في الفضاء الحلقي الالتفافي التقريبي. وكذلك ناقشنا المثاليات الالتفافية الأولية (الأولية تماما) الخشنة. وقدمنا عدة خصائص للمثاليات الالتفافية الاعظمية والاولية (الأولية تماما) الخشنة في الفضاء الحلقي الالتفافي التقريبي.