# Complete vector field on an open ball

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**Abstract:** -The theorem concerned in this paper has been proved before by the author, but in this paper the author shall give a new proof using the idea of lipschitzian method which give us a lot of results to formulate.

## 1. Introduction:-

The domain of real cartan triple factors Hilbert balls play a distinguish role: there gauge function can let the JB- norm of several different red JB- triple factors. The later fact seems to be one of the main, obstructs on the way to pure real geometric theory of JB- triples, and it is commonly agreed that a deep understanding of the structure of the complete real polynomial vector fields of Hilbert balls can be curtail in this direction.

Mangasarion and from Ovitz, are shown to have natural extensions valid when the mappings are only lips chizz continuo-us. Involved in these extensions is a compact, convex set of linear mappings called the general zed derivative, which can be assigned to any lips chit Zion continuous mapping and point of its (open) domain and which reduces to the us used derivative whenever the mapping is continuously differentiable.

## 2. The main result:

**2.1 Definition:** let a function f: $R^n \to R^n$we may identify f(x) $\frac{\partial}{\partial t}$with f. The vector ⊂ field f(x) $\frac{\partial}{\partial t}$ is said to be complete in B= {x:<x . x>< 1};B $R^n$ if for Rvery $x_0 \in$ B there is differentiable function t : IR → B where x(0) = $x_0$ and $\frac{\partial}{\partial t} f = f(x(t))$ t ∈ R.(BenYousif, 2005, pp1−10)

**2.2 Theorem:** Each tangent polynomial vector field belongs to the lie−algebra generated by the field $P_a$,$Q_a$ , ( a ∈ H) where Pa: x →a − <**x**,**a**> x . Where

Qa: x → a− [2<x,a>−<**x**,**x**>a].(Willard, 1968 & Hochschild, 1965)

**Proof:** Let Pa: x →a−<**x**,**a**>x so

As known that $\{abc\}_{spin}$= <**a**,**b**> c+<a,b>a−<**a**,**c**> b

Qa: x→a− $\{xax\}_{spin}$= a− 2<**x**,**a**> x+ <**x**,**x**>a.

And $\{abc\}_{\mathbf{Hill}}$ = $\frac{1}{2}$<**a**,**b**> c+ $\frac{1}{2}$<**a**,**b**> a,pa: x →a−{xax}$_{\mathbf{Hill}}$ then Qa= 2Pa

Pa = a−<**x**,**a**> x and Q = a−2<**x**,**a**>x + <**x**,**x**>a Qa−Pa = − <**x**,**a**> x+ <**x**,**x**> a.

{a b x}$_{\mathbf{Hill}}$ = $\frac{1}{2}$<**a**,**b**> x + $\frac{1}{2}$<**x**,**b**>a.

{b a x} $_{\mathbf{Hill}}$ = $\frac{1}{2}$<b, a> x + $\frac{1}{2} < x, a > b$ (a o b – b o a) x = $\frac{1}{2}$< x,b> a − $\frac{1}{2} < x, a > b$

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Qa−Pa=<x,x>a − <x,a> b =2(a o x − x o a) x 2(a o b – b o a) = <x,b> a − <x,a> b

Pa= a−<x,a> x and Qa= a− 2<x,a> x+ <x,x> a 2 Pa−Qa = a−<x,x>a = (1−<x,x>) a

Span {Pa,2Pa – Qa}∋ (Qa – Pa)

Span {Pa,Qa} = span { Pa,(1−<x,x>)a} . Then Qa−Pa = −(2Pa – Qa) + Pa =Qa – 2Pa. That's mean 2(a o x – x o a) x = − (1−<x,x>)a + Pa

Pa− (1−<x,x>) a =a − <x,a>x – a +<x,x>a = <x,x>a −<x,a>x , .

[P, Q] a Which Complete the theorem.

**2.3 Theorem** Let $p^-$ ∈Pol / IR$^n$, IR$^n$). Then the vector field $p^-$(x)$\frac{\partial}{\partial x}$ is

Complete in the unit ball $B^-$ if $p^-$ $is$ finite linear Compination of the Mappings $R^-$ (x), <$R^-$(x), x> x, (1−<x,x>)Q(x) Where Q, $R^-$ ∈ pol(R$^n$, R$^n$) .(Benyousif, 2004 &stacho, 2001)

**Proof:** let P denoted the set of all polynomials $p^-$ ∈Po (IR$^n$, IR$^n$) such that the vector field $p^-\frac{\partial}{\partial x}$is Complete in B. Since B is (real analytic sub manifold of IR$^n$ with analytic boundary ∂B, for a polynomial $p^-$ ∈ pol (IR$^n$, IR$^n$) we have:

$p^-$ ∈ P if $p^-$is tangent to∂B ,That is p={$p^-$ ∈ $pol(\mathrm{IR}^n, \mathrm{IR}^n)$$p^-$($x_1$,$x_2$,.....$x_n$)⊥($x_1$,$x_2$,.....$x_n$ ), $p^-$⊥e , for $x_1$,$x_2$,.....$x_n$∈ IR and $x_1^2 + x_2^2 + \cdots x_n^2 = 1\}$.

Now give any polynomial mapping (A) such that A: IR$^n$→ mat$^{(-)}$ (N,iR) we have < xA (x) . x>= < x, xA(x)$^T$> = < x,x (−A(x))> = − <xA(x), x>.

Thus necessarily <xA(x), x> = 0 that $xA(x)\perp x$ on the whole space $IR^n$ in particular $e\ A(e) \perp e$ $\forall$ unit vector which mean that $x \to xA(x)$ is complete polynomial vector field of second degree in unit spheres = $\sigma B$ S= $\sigma B$ = $(x_1^2 + x_2^2 + \cdots x_n^2)$ =1 <x,x>.

Conversely: let v be complete polynomial vector field on $\sigma B$ = s. we know that v(x) $=\sum_{k=1}^{n} v_k(x)\,, V_k(x)$ .for Some scalar–valued polynomials $v_k$: $IR^n \to R$ with the fundamental vector fields $V_k(x) = e_k - \langle e_k, x\rangle = e_k - x_k \sum_{i=1}^{n} x_i e_i$ .

Since the function $[1-(x_1^2 + \ldots + x_n^2)]$ vanishes on $\partial$B= s the vector field:–

$V^-(x) = v(x) - [1-(x_1^2, + \ldots + x_n^2)] \sum_{k=1}^{n} v_k(x)\, e_k$ .coincides with on the Spheres. However, with the standard matrices $E_{ik}$ with 1 at (i, k)–entry and o else where we can write: $V^-(x) = \sum_{k=1}^{n} v_k(x)\, [e_k - x_k \sum_{i}^{n} xi\ ei] - [\ 1-(X_1^2, + \ldots + x_n^2)] \sum_{k=1}^{n} v_k(x)\, e_k = \sum_{k=1}^{n} v_k(x) \sum_{i=1}^{n} x_i\, [x_i e_{k-} x_k e_i] = \sum_{k=1}^{n} v_k(x) \sum_{i=1}^{n} x_i x_k\, [E_{ik} - E_{ki}]$= x A(x) where , A(x) $=\sum_{1\le i<k\le n} (x_i v_k(x) - x_k v_i(x))[Ei_{k-} E_{ki}]$ is a polynomial map from $R^n$ in to mat $^{(-)}$ (N,R).

Now $P^-$= xA(x)– < xA(x), x> x + (1– < x,x>) Q(x).

$<P^{-}(e), e> = < (eA(e) - <eA(e), e>e + (1 - <e,e>) Q(e)), e>$.

$<P^{-}(e), e> = 0$. Which completethe proof.

**Remark:** let $V_k$: $x \to e_k - <e_k, x> x$ with , k=1,2,……n, then every complete polynomial vectc · field on the sphere s = $\partial B$, where B an open unit ball ($S = \partial B = \sum_{i=1}^{N} x_i^2 = 1$) Coincides with some vector field of the form $v(x) = \sum_{k=1}^{n} p_k(x) v_k(x)$ when restricted to S where $p_1, \ldots, p_n$: $R^N \to R$ are appropriate polynomials.

Example 1 : Although no non vanishing continuous vector field exists on sphere $s^2$ there are three mutually per appendicular vector field on $s^3 \subset R^4$ , that is a frame fields, let $s^3 = \{ (x_1, x_2, x_3, x_4), \sum_{i=1}^{4} x_i^2 = 1 \}$ and let vector field be given by:

$$X = -x_2\frac{\partial}{\partial X_1} + x_1\frac{\partial}{\partial X_2} + x4\frac{\partial}{\partial X_3} - x_3\frac{\partial}{\partial X_4} .$$

$$y = -x_3\frac{\partial}{\partial X_1} - x_4\frac{\partial}{\partial X_2} + x_1\frac{\partial}{\partial X_3} - x_2\frac{\partial}{\partial X_4} .$$

$$z = -x_4\frac{\partial}{\partial X_1} + x_3\frac{\partial}{\partial X_2} - x_2\frac{\partial}{\partial X_3} + x_1\frac{\partial}{\partial X_4} .$$

At p= ($x_1$, x2, x3, x4). Since at each point these are mutually Orthogonal Unit vectors in $IR^4$, they are independent. To see that they tangent to s it's enough to take inner product with x radius vector from the original to the point ones,

$<x^{\rightarrow}$,x> $=0$ $<z^{\rightarrow}$,x> = 0) =0

$<y^{\rightarrow}$, x> =0.

**Example2:** Assume v $\epsilon$ Pol ($IR^3$, $IR^3$) and< v (e).e =0 for <e,e> =1 <v(e).e>=0

mean that v is tangent to unit – sphere.

Let a = v(0),w=v−va then w,also tangent to the unit−Spherebut already w(0)=0

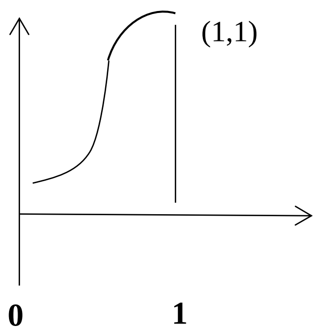
$e_{v\varphi}=\begin{pmatrix}\cos v\\ sinv\ cos\phi\\ sinv\cos\phi\end{pmatrix}$Then< w $(e_{v\varphi}), e_{v\varphi}>= 0.$

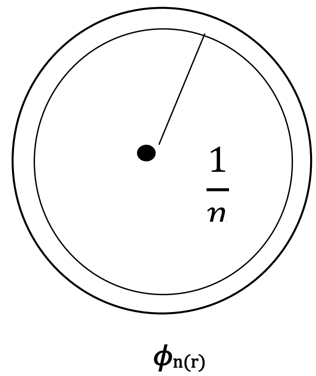
$\frac{\partial}{\partial u}\frac{\partial}{\partial v}$< w($e_v\varphi$) , $e_v\varphi$> = 0 ,$\frac{\partial}{\partial u}$ ($e_v\varphi$) = $\begin{pmatrix}0\\ -sinv\ sin\varphi\\ \sin v\ cos\varphi\end{pmatrix}$

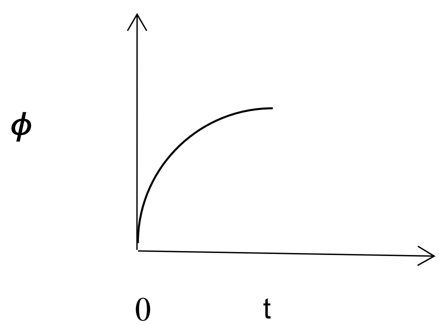
$\frac{\partial}{\partial u}$ ($\frac{\partial}{\partial u}$ ($e_v\varphi$)) $=\begin{pmatrix}0\\ -cosv\ sin\varphi\\ \cos v\ cos\varphi\end{pmatrix}$

**Example3:**

P:B ($R^n$) → $R^n$ complete analytic filed$[\,1-<\ x,x\ >]^{\frac{1}{n}}$P(x)= $P^n$(x) .




$$\Psi\ (\exp\ [-1/\ (nr)^2]) = \phi_n\ (r) \qquad \Psi\ (t) = 1 - \exp\ [1 - 1/(1 - t^2\ )]$$

$$P_n\ (x) = \phi_n\ (x^2 + y^2)\ p\ (x)$$

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