Complete and Analytic Vector Field On R^N

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Abstract:-

In this paper the author shall give proofs of three of most important theorems on complete polynomial and analytical vector fields.

1. Introduction:-

During my work on polynomial vector fields we proved some Theorems and lemmas which describes how looks like the behavior of complete vector fields with its domain, in this paper we give some Theorems ,lemmas and examples with graphs to show how clean our results .

The theorem of analytic function will be the key of analytical vector fields on an open Euclidean ball subset fromR^N .

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2. Theorem.

Let $f: R^N \to R$ be a polynomial such that $f(x) = 0$ for $x \in S$ where$S = (\langle x, x\rangle = 1)$. Then there exists a polynomial$Q: R^N \to R$, such that $f(x) = (1 - \langle x, x\rangle)Q(x)$.

Proof. Let $g: B \to R$ be function on the unit ball$B: (\langle x, x\rangle < 1)$, defined by$g(x) = \frac{f(x)}{(1-\langle x,x\rangle)}$, the function g is analytic, since it is the quotient of two polynomials. (Benyousif, 2004)

Thus $g(x) = \sum_{k=0}^{\infty} g_k(x)$ where g_k are $k-$homogeneous polynomial onR^N. We have $f(\pm e) = 0$ whenever$\langle e, e\rangle = 1$, there exists a polynomial $P_e: R \to R$ of degree $\le \ degf - 2$ such that$(1 - t^2)P_e(t) = f(te)$. It follows that, for every fixed unit vector$e \in R^N$, $g(te) = \frac{f(te)}{(1-t^2)} = P_e(t) \sum_{k=0}^{degf-2} \alpha_k(e)t^k$with suitable constants$\alpha_0(e), \ldots, \alpha_{deg-2}(e) \in R$. Thus $g = \sum_{k=0}^{degf-2} g_k$is a polynomial. This completes the proof.

3. Theorem.

Let Ψ bounded continuously differentiable, $0 < \Psi \le 1$ and $W: B(R^N) \to R^N$ is a complete vector field then ΨW is complete vector field.

Proof. Let w (t) =$\int_{\mathcal{T}=0}^{t} \psi\ (x_{\mathcal{T}})d\mathcal{T}$, let $x_0 \in R^n, x_t: \frac{d}{dt}\, x_t = w(x_t)\ \forall\ t \in R, x_t\ well - defined\ because\ w\ is\ complete$.

$y_t = x_{w(t)}\ ;\ y_0 = x_{w(t)} = x_0$

$y'(t) = \frac{d}{dt} y(t) = \Psi(y(t))\, w(y(t))$

$y(t) = x(w(t))$; $x(0) = x_0$

$x'(t) = w(x(t))$

$y'(t) = \frac{d}{dt} y(t) = x'\big(w(t)\big)w'(t) = w\left(x\big(w(t)\big)\right).w'(t) = w(y(t)).w'(t)$.

$\Psi(y(t)) = w'(t)$; $w(0) = 0$

$\Psi(x(w(t))) = w'(t)$

$\varphi = \Psi o x$ Bounded By assumption (also C^1 smooth)

$w'(t) = \varphi(w(t))$; $w(0) = 0$

$$\frac{\dot{w}(t)}{\varphi(w(t))} = 1 \quad , \qquad \Phi(s) = \int_{\tau=0}^{s} \frac{d\tau}{\varphi(\tau)}$$

$\frac{d}{dt}\Phi\big(w(t)\big) = 1$; $\Phi(0) = 0$ Monotonic increasing strictly imply that Φ^{-1} exists $\Phi(w(t)) = t$ then $w(t) = \Phi^{-1}(t)$

Suite our requirement

4. Theorem.

If P: $IR^N \rightarrow IR$ is a polynomial function and $0 \neq \emptyset$:

$IR^k \rightarrow IR$ is an affine function such that P (q) = 0 for the points q of the hyper plane {q $\in IR^N$: $\emptyset$(q) = 0} then $\emptyset$ is a divisor of P in the end sense that P = $\emptyset$Q with some (unique) polynomial Q: $IR^N \rightarrow IR$.

Proof. Trivially, any two hyper planes are affine images of each other. In particular there is a one–to–one affine (i.e. linear + constant) map– ping A: $IR^N \leftrightarrow IR^N$, such that {q $\in IR^N$: ø (q) = 0} = A ({q$\in$ IR^N: X_1(q) = 0}). Then R: =PoA is a polynomial function such that R (q) =0 or the points of the hyper plane {q $\in$ IR^N: x_1(q) =0}. (Benyousif, 2004 & stacho, 2001)

We can write $R = \sum_{k_1 \ldots k_N=0}^{d} \alpha_{K1 \ldots\ldots\ldots, K_N}\, x_1^{k1} \ldots x_N^{kN}$ with a suitable finite family of coefficients $\alpha_{K1 \ldots KN}$by the Taylor formula,$\alpha_{K_1 \ldots\ldots\ldots K_N} = \frac{\partial^{k1+\cdots+k_N}}{\partial {x_1}^{k1} \ldots \partial x_N^{k_N}}\bigg|_{x_1 \ldots x_N=0} R$.

It follows $\alpha_{k_1 \ldots\ldots\ldots k_N} = 0$ for$k > 0$, since R vanishes for $x_1 = 0$. This means that $R = x_1 \text{o}\, R$ with the polynomial

$R_0 := \sum_{k1=1}^{d} \sum_{k2,\ldots\ldots kn=0}^{d} x_1^{k1-1} x_2^{k2} \ldots \ldots \ldots \ldots x_N^{k_N}$. By the same argument. That is o is the sum of a linear functional with a constant.

Applied for the polynomial function ∅ of degree d=1 in place of R, we see that∅ o A= αx_1 for some constant (polynomial of degree 0)$\alpha \neq 0$. That is ∅ = αx_1 o A^{-1}. Therefore

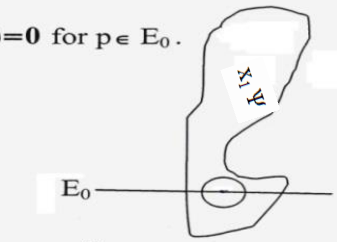
P = Ro A^{-1} = [x_1 R_0] o A^{-1} = (x_1 o A^{-1})=(R_0 o A^{-1}) = ∅($\frac{1}{\alpha}$$R_0$o A^{-1}). Since the inverse of an affine mapping is affine as well, the function

Q: = (1/α R_0oA^{-1}) is a polynomial which suits the statement of theorem.

5. Theorem:

Assume that $G \subset R^n$ is an open connected set such that $G \cap E_0 \neq \phi$ and let ϕ: $G \rightarrow R$ be analytic function such that ϕ (x) = 0 for all $x \in G \cap E_0$. Then (x) = $x_1 \Psi$ for some analytic function Ψ : $G \rightarrow R$ where $x_1 = < x.e >$ and x = (x_1,

Proof: Let $E_0 = < p \in R^N : x_1(p) = 0 >$, be a hyper-plane $\phi(p)=0$ for $p \in E_0$.



$$\phi(p)=\sum_{k=1}^{\infty} \sum_{n_1+\dots+n_N=h} a_{n_1 \dots n_N} x_1^{n_1}(p) \dots x_N^{n_N}(p)$$

$$p \in E_0 \Longrightarrow x_1(p)=0, \quad x_1^{n_1}(p) \dots x_N^{n_N}(p)=0, \text{ if } n_1>0$$

$$0=\phi(p)=\sum_{k=1}^{\infty} \sum_{n_2+\dots+n_N=k} a_{n_0 n_2 \dots n_N} x_2^{n_2}(p) \dots x_N^{n_N}(p), \text{ by assumption.}$$

$$p:=\xi_2 e_2+\dots+\xi_N e_N \in E_0, \quad \xi_2, \dots, \xi_N \in R, \text{ arbitrary.}$$

$$0=\phi(p)=\sum_{n_2+\dots+n_N=k} a_{0 n_2 \dots n_N} \xi_2^{n_2} \dots \xi_N^{n_N}$$

$$a_{0 n_2 \dots n_N}=\frac{\partial^{n_2+\dots+n_N} \phi(\xi_2 e_2+\dots+\xi_N e_N)}{\partial x_2^{n_2} \dots \partial x_N^{n_N}} \frac{1}{n_2! \dots n_N!}=0.$$

$$a_{0 n_2 \dots n_N} = 0, \forall n_2, \dots, n_N.$$

$$\phi(p)=\sum_{k=1}^{\infty} \sum_{\substack{n_1+\dots+n_N=k \\ n_1>0}} a_{n_1 n_2 \dots n_N} x_1^{n_1}(p) \dots x_N^{n_N}(p)$$

$$=x_1(p) \sum_{k=1}^{\infty} \sum_{\substack{n_1+\dots+n_N=k \\ n_1>0}} a_{n_1 n_2 \dots n_N} x_1^{n_1-1}(p) \dots x_N^{n_N}(p)$$

$$=x_1(p)\, \psi(p).$$

$$\psi(p)=\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n_2+\dots+n_N=l} a_{n_1 n_2 \dots n_N} x_1^{m}(p) \dots x_N^{n_N}(p)$$

With $m=n_1-1$ and $l=k-1$.

$x_2, \ldots\ldots x_n) \in R^n$.

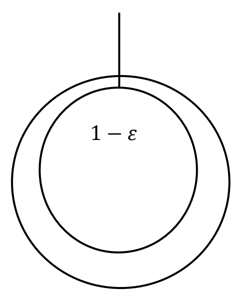
Which complete the proof of most important theorems in analytic vector field on a unit ball subset from R^N complete

Proposition: By theorem $\varphi_n V$ is complete analytic vector filed on $B(R^N)$. (Walker, 1950)

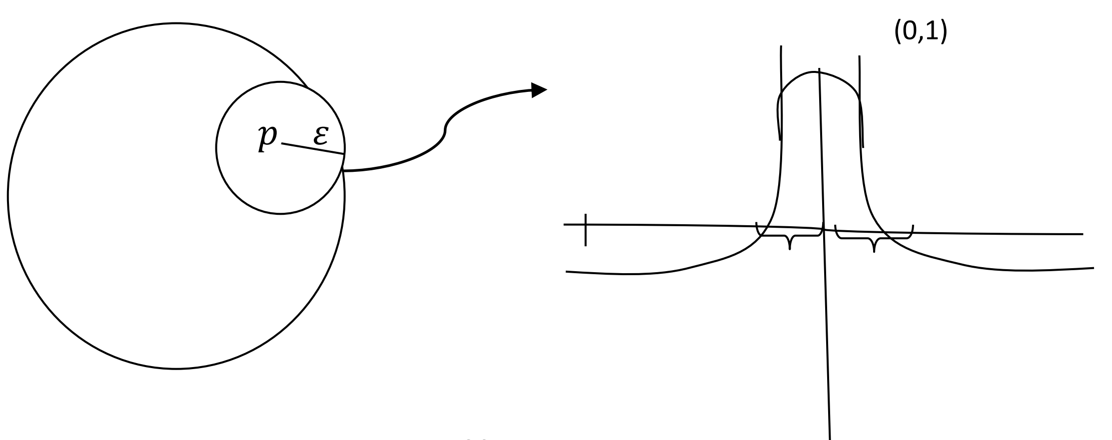
Let $V(x) = \sum_{k=0}^{\infty} \sum_{k_1 \ldots k_N = k} a_{k_1 \ldots k_N} x_1^{k_1} \ldots x_N^{k_N}$ Taylor series

If $\forall\ \varepsilon > 0 \quad V^{[n]}(x) := \sum_{k=0}^{N} \sum_{k_1 \ldots k_N = k} a_{k_1 \ldots k_N} x_1^{k_1} \ldots x_N^{k_N} \underset{\rightarrow}{\overset{\rightarrow}{}} \ V$

Uniform on $(1-\varepsilon)\ B(R^N)$



Then $\varphi_N V^{[N]} \underset{loc}{\overset{\rightarrow}{\rightarrow}} \ V$ complete polynomial vector filed.



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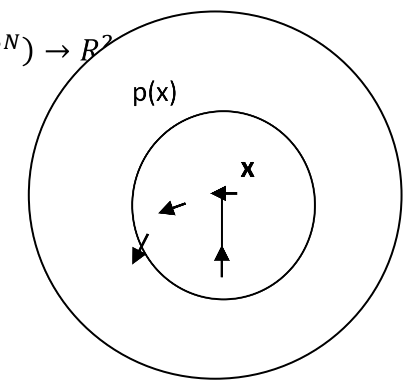
ε p ε

$$\varphi_{p,\varepsilon} = 1 - \langle \frac{1}{\varepsilon}(x-p), \frac{1}{\varepsilon}(x-p) \rangle^{\mathrm{n}_{\mathrm{p},\varepsilon}}$$

Example

$$P: B(R^N) \to B^2$$

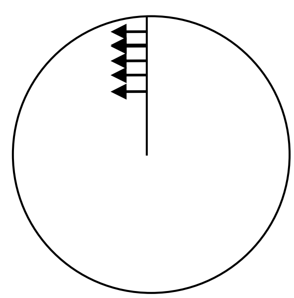


$$(r.cos\varphi, r.sin\varphi) \to \left(-t^2 g\left(r\frac{\pi}{2}\right).sin\varphi, t^2 g\left(r\frac{\pi}{2}\right).cos\varphi\right)$$

$$\binom{x}{y} \to \binom{-y}{x}$$

$$t^2 gr = \sum_{k=0}^{\infty} \alpha_k r^{2n}$$

$$\mathrm{P}\left(\binom{x}{y}\right) := \sum_{k=0}^{m} \alpha_k ((x^2+y^2)^n) \binom{-y}{x}$$



Locally uniform, Approximation.

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